Combinatorial Synergies in Nine Pictures

Exercises prepared by *Bernd Sturmfels*

Below is one question for each picture on the poster for the DRG Priority Program SPP 2458. We shall discuss these at the Kick-Off Meeting in Osnabrück, September 11-13, 2024.

1. *Enumeration*: Draw Young's poset for all partitions that fit into a 3×3 box. How many maximal chains does this poset have? Compute the zeta polynomial.

2. *Statistics*: Our experiment is to toss a biased coin three times, so it has four outcomes: HHH, HHT, HTT, TTT. We sample data from 1000 repetitions of this experiment. Draw the model, your data and their maximum likelihood estimate in a tetrahedron.

3. *Commutative Algebra*: The picture shows a monomial ideal in three variables. Find its minimal generators. Compute a free resolution and the irreducible decomposition.

4. *Mathematical Physics*: The positive Grassmannian $\mathrm{Gr}(3,6)_{>0}$ is homeomorphic to the open orthant $\mathbb{R}^9_{>0}$. Write down an explicit bijection. Can you use the plabic graph?

5. *Dynkin Classication*: How many hyperplanes are in the reflection arrangement of type E_6? Write down the root system explicitly. What is the order of the Weyl group?

6. *Convexity*: How many monotone edge paths are there between two antipodal vertices of the permutohedron. What is the combinatorial structure on the set of these paths?

7. *Lattice Points*: The product of three triangles is a 6-dim'l polytope $\Delta_2 \times \Delta_2 \times \Delta_2$. Compute its Ehrhart polynomial. What does it mean in Statistics or Commutative Algebra?

8. *Matroids*: Determine the matroid polytope of the Fano matroid. Find a Lorentzian polynomial with this Newton polytope. Determine the Chow ring of this matroid.

9. *Nonlinear optimization*: 4×4-correlation matrix are symmetric, positive semi-definite, with 1's on the diagonal. They form a 6-dimensional convex body. Determine its faces.

Bonus Question: Which career paths are envisioned by Combinatorial Synergies members?