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Cluster Algebras
Exercises

(April 2, 2025)

Stars indicate difficult questions.

Exercise 1

Let $c \in \{2, 3\}$, consider $\mathcal{A}(1, c)$.

(i) Describe the cluster variables you get in these cases.

(ii) If you associate $x_1 \leftrightarrow -\alpha_1$, $x_2 \leftrightarrow -\alpha_2$ and positive roots for denominators as in Remark 1.2(3), what root system do you get?

Exercise 2

Prove the following result: Every cluster variable x_m in $\mathcal{A}(b, c)$ is an element of the ring of Laurent polynomials $\mathbb{Z}\left[x_1^{\pm 1}, x_2^{\pm 1}\right]$.

Exercise 3

($\star\star$) Consider $\mathcal{A}(b, c)$ for $bc \geqslant 4$ ($b, c \in \mathbb{Z}_{\geqslant 0}$). Argue that one gets infinitely many cluster variables.

Exercise 4

Show that there is a bijection between cluster quivers with n mutable and $m - n$ frozen vertices and extended skew-symmetric matrices of size $m \times n$.

Exercise 5

Consider the quiver $Q: \ 1 \longrightarrow 2 \longrightarrow 3$ with three mutable vertices. Work out $\mu_1(\mu_2(Q))$, $\mu_3(\mu_2(Q))$, $\mu_1(Q)$ and $\mu_3(Q)$.

Exercise 6

Consider the quiver

$$Q: \quad \begin{tikzcd} 1 \arrow[rr, "3"] & & 2 \arrow[dl, "3"] \\ & 3 \arrow[ul, "3"] & \end{tikzcd}$$

(i) Determine the mutated cluster quivers $\mu_i(Q)$ where $i \in \{1, 2, 3\}$.

(ii) Determine the mutated cluster quivers $\mu_i(\mu_1(Q))$ where $i \in \{1, 2, 3\}$.

(iii) Describe the quivers one obtains by applying further mutations. Can you come up with a pattern?

Exercise 7

Prove that mutations are involutive *i.e.* for any quiver Q and k a mutable vertex in Q, we have $\mu_k(\mu_k(Q)) = Q$.

Exercise 8

Consider the following quiver.

$$Q : \begin{tikzcd} \boxed{6} & & \boxed{7} \arrow[dl] & & \boxed{3} \arrow[dl] \\ & 1 \arrow[ul] \arrow[rr] & & 2 \arrow[ul] \arrow[dr] & \\ \boxed{5} \arrow[ur] & & & & \boxed{4} \end{tikzcd}$$

Compute the clusters, cluster variables and cluster algebra of Q.

Exercise 9

Consider the following quiver.

$$Q : \begin{tikzcd} 1 \arrow[rr, "2"] & & 2 \arrow[dl, "2"] \\ & 3 \arrow[ul, "2"] & \end{tikzcd}$$

(i) Compute $\mu_i(Q)$ and their corresponding exchange relations for $i \in \{1, 2, 3\}$. Comment on your result.

(ii) Specialise the initial cluster (x_1, x_2, x_3) to $x_i = 1$ for $i \in \{1, 2, 3\}$. Use the exchange relations to find the specialised values of other clusters. Do several mutations.

(iii) ($\star$) Show that for any cluster, its corresponding specialized values (a_1, a_2, a_3) satisfy

$$a_1^2 + a_2^2 + a_3^2 = 3a_1a_2a_3.$$

Remark 1. Every such triple is therefore a Markov triple.

Exercise 10

Let (x, Q) be a seed and take $n = m \geq 3$. Consider i and j distinct mutable vertices.

(i) Compute the i^{th} cluster variable after applying mutations μ_i, μ_j and μ_i again.

(ii) Argue that the outcome is an element of $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

(iii) What can you say about the coefficients?

Exercise 11

Let Q be the Kronecker quiver:

$$Q: \; 1 \xrightarrow{2} 2$$

Check that the cluster algebra $\mathcal{A}(Q)$ is not of finite type but that it is of finite mutation type. You can use Exercise 4.