Spring School “The Dynkin Classification” 2025
Ruhr-Universität Bochum

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Coxeter groups and root systems
Exercises

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Exercise 1 (★☆☆)

Let $w \in O(V)$ be an element in the orthogonal group and let $\alpha \in V \setminus \{0\}$. Show that

$$s_{w(\alpha)} = w s_\alpha w^{-1}.$$

Exercise 2 (★☆☆)

Let Φ be a root system of rank 2.

Show that $W(\Phi)$ is a dihedral group meaning that it is the group of symmetries of a regular polygon.

Exercise 3 (★☆☆)

Let W of type $I_2(6)$ be the symmetry group of the regular hexagon.

Find a Coxeter system of rank 3 that is isomorphic to $I_2(6)$ as an abstract group.

Exercise 4 (★☆☆)

Given the property that the simple roots Δ form a basis of span(Φ), prove that $\langle \alpha \mid \beta \rangle < 0$ for $\alpha, \beta \in \Delta, \alpha \neq \beta$.

Exercise 5 (★☆☆)

Prove that any two expressions for an element $\sigma \in \mathfrak{S}_n$ written in terms of simple transpositions have the same parity.

Exercise 6 (★☆☆)

Show that for $\sigma \in \mathfrak{S}_n$, the length is given by

$$\ell(\sigma) = |\{i < j \mid \sigma(i) > \sigma(j)\}|\,.$$

Exercise 7 (★☆☆)

A (standard) Coxeter element in a reflection group W is a product of the simple generators in any order.

Prove that all Coxeter elements are conjugate in W.

Exercise 8 (★★☆)

Determine the number of (standard) Coxeter elements for all finite irreducible reflection groups.

Hint: Compute these numbers for types A_2, B_2, A_3, B_3, H_3 to first guess a formula.

Exercise 9 (★☆☆)

Compute the eigenvalues for

(a) the long cycle $(1, 2, 3, \dots, n) \in \mathfrak{S}_n$, and

(b) the long cycle $(1, 2, 3, \dots, n, -1, -2, \dots, -n) \in W(B_n)$.

Exercise 10 (★☆☆)

Let (W, S) be a finite Coxeter system with $S = \{s_1, \dots, s_n\}$. Show that the alternating group of all elements in W of even length is generated by $\{s_i s_n\}_{i=1}^{n-1}$.

Exercise 11 (★★☆)

Suppose W stabilises a lattice. Show that $m(\alpha, \beta) \in \{2, 3, 4, 6\}$ for all $\alpha, \beta \in \Delta$.

Exercise 12 (★★☆)

Show that $\Phi = \left\{ x \in \mathbb{Z}^3 \;\middle|\; \sum_{i=1}^{3} x_i = 0,\ |x|^2 \in \{2, 6\} \right\}$ is the root system of type $G_2 = I_2(6)$.

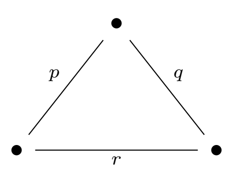
Identify a simple system $\Delta \subseteq \Phi$.

Exercise 13 (★★☆)

Show that $\Phi = \left\{ x \in \mathbb{Z}^4 \cup \mathbb{Z}^4 + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \;\middle|\; |x|^2 \in \{1, 2\} \right\}$ is the root system of type F_4.

Exercise 14 (★★★)

Let (W, S) be a Coxeter system of rank 3 with $S = \{s_1, s_2, s_3\}$ and with the following Dynkin diagram:



Let $d = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$. Show that the bilinear form $\langle \cdot \mid \cdot \rangle$ is

(a) positive definite if and only if $d > 1$, and

(b) positive semidefinite but degenerate if and only if $d = 1$.

Caveat: Do not use the classification of positive definite and of positive semidefinite graphs.

Exercise 15 (★★★)

For a finite reflection group W denote by

$$f_W(q) = \sum_{w \in W} q^{\ell(w)}$$

the length generating function. Find nice product formulas for f_W for

(a) $W = \mathfrak{S}_n$, and

(b) $W = W(B_n)$.

Compare your results for types A_n, B_n with the eigenvalues you computed in Exercise 9.

Remark. The polynomial f_W is called Poincaré polynomial of W and it always has a nice product formula in terms of the degrees of a set of algebraically independent homogeneous polynomials that generate the ring of W-invariant polynomials. Also, these numbers appear in general in the eigenvalues of the Coxeter elements.

Exercise 16 (★★☆)

Find the symmetry group for each Platonic solid.

Hint: Convince yourself that the symmetry group of a Platonic solid is a reflection group and then use the classification.

Exercise 17 (★★★★)

Let W be a finite reflection group. The descents of $w \in W$ are defined as those simple roots that are sent to negative roots, $-\Pi \cap w(\Delta)$.

Prove that the descent generating function

$$\mathrm{E}_W(q) = \sum_{w \in W} q^{|\operatorname{Des}(w)|}$$

has only real roots.

Caveat: For the symmetric group, this is the well-studied **Eulerian polynomial**. The real-rootedness is proven case-by-case using the classification in multiple papers, while the exceptional types are just checked using computers. It is an open problem to prove this property uniformly.

Have fun!