Lie algebras

Exercises

Exercise 1. Let $\varphi : L \to L'$ be a homomorphism of Lie algebras. Prove that $\mathrm{Ker}(\varphi)$ is an ideal of L and that $\mathrm{Im}(\varphi)$ is a Lie subalgebra of L'.

Exercise 2. Let L_1 and L_2 be Lie algebras. Show that the direct sum of vector spaces $L_1 \oplus L_2$ can be canonically endowed with the structure of a Lie algebra.

Exercise 3. Let $L = \mathfrak{sl}_2(\mathbb{C})$ and let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be an ordered basis for L. Compute the matrices of $\mathrm{ad}(e)$, $\mathrm{ad}(h)$ and $\mathrm{ad}(f)$ with respect to this basis.

Exercise 4. Let $n \geq 1$ and F a field. Prove that the trace $\mathrm{tr} : \mathfrak{gl}_n(F) \to F$ is a homomorphism of Lie algebras, where F is endowed with the trivial Lie bracket. Conclude that $\mathfrak{sl}_n(F)$ is an ideal of $\mathfrak{gl}_n(F)$.

Exercise 5. Let L be a Lie algebra. Show that the set of inner derivations of L forms an ideal in $\mathrm{Der}(L)$.

Exercise 6. Let F be a field with $\mathrm{char}(F) \neq 2$. Prove that $\mathfrak{sl}_2(F)$ is simple.

Exercise 7. Let V be an F-vector space of dimension n. Suppose that $x \in \mathfrak{gl}(V)$ has $\alpha_1, \ldots, \alpha_n \in F$ as distinct eigenvalues on V. Prove that the scalars $\alpha_i - \alpha_j$, $1 \leq i, j \leq n$, are the (not necessarily distinct) eigenvalues of $\mathrm{ad}(x)$ on $\mathfrak{gl}(V)$.

Exercise 8. Let

$$\mathfrak{n} = \{(a_{ij}) \in \mathfrak{gl}_3(\mathbb{C}) \mid a_{ij} = 0 \text{ for } 1 \leq j \leq i \leq 3\}$$

and

$$\mathfrak{b} = \{(a_{ij}) \in \mathfrak{gl}_3(\mathbb{C}) \mid a_{ij} = 0 \text{ for } 1 \leq j < i \leq 3\}.$$

(a) Show that $\mathfrak{n}$ is nilpotent.

(b) Show that $\mathfrak{b}$ is solvable but not nilpotent.

Exercise 9. Let L be a Lie algebra. Prove the following assertions.

(a) If L is nilpotent, then so is every subalgebra of L and also every homomorphic image of L.

(b) If $L/Z(L)$ is nilpotent, then so is L.

Exercise 10. Let L be a simple Lie algebra. Prove that L is isomorphic to a linear Lie algebra.

Exercise 11. Let L be a Lie algebra with maximal toral subalgebra $H \subseteq L$. Let $\alpha, \beta \in H^*$.

(a) We have $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ for the root spaces.

(b) If $x \in L_\alpha$, $\alpha \neq 0$, then $\mathrm{ad}(x)$ is nilpotent.

Exercise 12. Let L be a Lie algebra and let κ be the Killing form of L.

(a) The form κ is *associative* in the sense that $\kappa([x,y],z) = \kappa(x,[y,z])$ for all $x,y,z \in L$.

(b) Let $H \subseteq L$ be a maximal toral subalgebra and let $\alpha, \beta \in H^*$ be roots of L with $\alpha + \beta \neq 0$. Then $\kappa(L_\alpha, L_\beta) = 0$.