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Spring school: Dynkin classification
Day 5: Simple singularities

Exercise 1.

(1) Show that

$$\mathrm{SU}_2(\mathbb{C}) = \{\left(\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\right) : |\alpha|^2 + |\beta|^2 = 1\}$$

is the subgroup of $\mathrm{SL}_2(\mathbb{C})$ consisting of elements letting invariant the standard Hermitian form $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1\bar{y}_1 + x_2\bar{y}_2$ on $\mathbb{C}^2$. The group $\mathrm{SU}_2(\mathbb{C})$ is called the *complex special unitary group* (of dimension two).

(2) Show that any finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ is conjugate to a subgroup of $\mathrm{SU}_2(\mathbb{C})$.

(3) Show that $\mathrm{SU}_2(\mathbb{C})$ contains exactly one element of order 2.

Exercise 2. (Singularities of curves) Let X be the nodal curve given by the equation $y^2 = x^3 + x^2$.

(1) Find a surjective (polynomial) map $\mathbb{A}^1 \to X$.

(2) Let $P \in X$ be the singular point. Show that the completed local ring $\widehat{\mathcal{O}}_{X,P}$ of X at P is isomorphic (as $\mathbb{C}$-algebra) to $\mathbb{C}[\![x, y]\!]/(xy)$.

(3)* Can you construct (many) more further plane curve singularities (and isomorphisms between some of them)?

Exercise 3. (Du Val singularity of type A_{n-1}) Let $n \geq 2$ and let ζ be a primitive complex root of unity of order n. Let $X = \mathbb{A}^2/G$, where $G = \langle\gamma\rangle \cong \mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{A}^2$ via $\gamma.x = \zeta x$, $\gamma.y = \zeta^{-1}y$. Let $P = (0, 0) \in X$ be the singular point. Show that

$$\widehat{\mathcal{O}}_{X,P} = \mathbb{C}[\![x, y]\!]^G \cong \mathbb{C}[\![u, v, w]\!]/(u^2 + v^2 + w^n).$$

Exercise 4. (Du Val singularity of type D_n) Let $n \geq 2$. Let $G = \mathbb{D}_n$ be the binary dihedral group acting on $\mathbb{C}^2$ (as in the lecture). Let $X = \mathbb{C}^2/G$ and let $P = (0, 0) \in X$ be the singular point. Show that

$$\widehat{\mathcal{O}}_{X,P} = \mathbb{C}[\![x, y]\!]^G \cong \mathbb{C}[\![u, v, w]\!]/(u^{n+1} + uv^2 + w^2).$$

(The full proof as in Exercise 2 might be lengthy. We do not require it.)

Exercise 5. (Blow up of the nodal curve) Let $X = \{y^2 = x^3 + x^2\} \subseteq \mathbb{A}^2$ be the nodal curve. Let $P \in X$ be the origin. Let $\mathrm{Bl}_P(\mathbb{A}^2) = \{(x, y), [u : v] \in \mathbb{A}^2 \times \mathbb{P}^1 : xv = yu\}$ be the blow up of $\mathbb{A}^2$ at P. Compute the strict transform $\tilde{X}$ of X inside $\mathrm{Bl}_P(\mathbb{A}^2)$.

Exercise 6. (Blow-up Du Val singularities) Compute the blow-ups and the dual graphs of the Du Val singularities of type A_2, A_3 and D_4.

Exercise 7. Prove that the singularities of type A_3 $(x^2 + y^2 + z^4 = 0)$ and D_3 $(x^2 + y^2z + z^2 = 0)$ are isomorphic.