Solutions to the exercises on du Val singularities

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Exercise 1. 1. Show that

$$\mathrm{SU}_2(\mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \,\middle|\, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

is the subgroup of $\mathrm{SL}_2(\mathbb{C})$ consisting of elements letting invariant the standard Hermitian form

$$\begin{gathered} (\ ,\ )\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C} \\ \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto x_1\overline{y}_1 + x_2\overline{y}_2 \end{gathered}$$

The group $\mathrm{SU}_2(\mathbb{C})$ is called the complex special unitary group (of dimension two).

2. Show that any finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ is conjugate to a subgroup of $\mathrm{SU}_2(\mathbb{C})$.

3. Show that $\mathrm{SU}_2(\mathbb{C})$ contains exactly one element of order 2.

Proof. 1. For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we calculate

$$\begin{aligned} \left(\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}, \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix} \right) &= (ax_1 + bx_2) \cdot \overline{(ay_1 + by_2)} + (cx_1 + dx_2) \cdot \overline{(cy_1 + dy_2)} \\ &= a\overline{a}x_1\overline{y}_1 + a\overline{b}x_1\overline{y}_2 + b\overline{a}x_2\overline{y}_1 + b\overline{b}x_2\overline{y}_2 + c\overline{c}x_1\overline{y}_1 + c\overline{d}x_1\overline{y}_2 + d\overline{c}x_2\overline{y}_1 + d\overline{d}x_2\overline{y}_2 \\ &= (a\overline{a} + c\overline{c})x_1\overline{y}_1 + (a\overline{b} + c\overline{d})x_1\overline{y}_2 + (b\overline{a} + d\overline{c})x_2\overline{y}_1 + (b\overline{b} + d\overline{d})x_2\overline{y}_2 \\ &\overset{!}{=} x_1\overline{y}_1 + x_2\overline{y}_2 \end{aligned}$$

From this calculation, it is clear that if $A \in \mathrm{SU}_2(\mathbb{C})$, then $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{C}^2$. Setting $x_1 = y_1 = 1$ and $x_2 = y_2 = 0$ gives us $a\overline{a} + c\overline{c} = |a|^2 + |c|^2 = 1$ and analogously we get $|b|^2 + |d|^2 = 1$. Setting $x_1 = y_2 = 1$ and $x_2 = y_1 = 0$ we get $a\overline{b} + c\overline{d} = 0$. Using the assumption $ad - bc = 1$, we multiply this with $\overline{a}$ to get:

$$\begin{aligned} \overline{a} &= a\overline{a}d - \overline{a}bc \\ &= (1 - c\overline{c})d + \overline{c}dc \\ &= d \end{aligned}$$

Analogously, we get $c = -\overline{b}$.

2. Let $G \subseteq \mathrm{SL}_2(\mathbb{C})$ be finite and consider $\mathbb{C}^2$ with the standard Hermitian form $(,)$. Define

$$\langle x, y \rangle \coloneqq \frac{1}{|G|} \cdot \sum_{g \in G} (gx, gy) \quad \forall x, y \in \mathbb{C}^2.$$

Then:

- $\langle x, y \rangle \geq 0$ for all $x, y \in \mathbb{C}^2$, since $(a, b) \geq 0$ for all $a, b \in \mathbb{C}^2$.

- If $\langle x, y\rangle = 0$, this implies that $x = 0$. This follows since the assumption implies that $(gx, gx) = 0$ for all $g \in G$ and hence $gx = 0$ for all $g \in G$ since $(,)$ is a Hermitian form. This then implies that $x = 0$.
- We calculate

$$\langle x, y\rangle = \frac{1}{|G|}\sum_{g\in G}(gx, gy) = \frac{1}{|G|}\sum_{g\in G}\overline{(gy, gx)} = \overline{\langle y, x\rangle}.$$

This implies that $\langle,\rangle$ is a Hermitian form. Now let $h \in G$. Then

$$\langle hx, hy\rangle = \frac{1}{|G|}\sum_{g\in G}(ghx, ghy) \overset{g' := gh}{=} \frac{1}{|G|}\sum_{g'\in G}(g'x, g'y) = \langle x, y\rangle.$$

Then, $G \subseteq \mathrm{SU}(\mathbb{C}^2, \langle,\rangle) := \left\{h \in \mathrm{SL}_2(\mathbb{C}) \,\middle|\, \langle hx, hy\rangle = \langle x, y\rangle \ \forall x, y \in \mathbb{C}^2\right\}$ and since all Hermitian forms are isomorphic, there is an isomorphism $\mathrm{SU}(\mathbb{C}^2, \langle,\rangle) \cong \mathrm{SU}_2(\mathbb{C})$ given by conjugation with a matrix S, which then implies that the conjugate of G is a (finite) subgroup of $\mathrm{SU}_2(\mathbb{C})$. Indeed, we can find a $\mathbb{C}$-linear isomorphism $S\colon \mathbb{C}^2 \xrightarrow{\cong} \mathbb{C}^2$ such that $\langle x, y\rangle = (Sx, Sy)$. Then, if $g \in \mathrm{SU}_2(\mathbb{C}, \langle,\rangle)$, then $SgS^{-1} \in \mathrm{SU}(\mathbb{C}^2, (,))$, by the following calculation:

$$(x, y) = \langle S^{-1}x, S^{-1}y\rangle = \langle gS^{-1}x, gS^{-1}y\rangle = (SgS^{-1}x, SgS^{-1}y).$$

3. We calculate
$$\begin{pmatrix}\alpha & \beta \\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}^2 = \begin{pmatrix}\alpha^2 - |\beta|^2 & \alpha\beta + \overline{\alpha}\beta \\ -\overline{\beta}\alpha - \overline{\beta\alpha} & \overline{\alpha}^2 - |\beta|^2\end{pmatrix} \overset{!}{=} \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$$
If we assume a matrix is of order 2, the above calculation implies that $\alpha^2 = \overline{\alpha}^2$ and hence $\alpha = \pm\overline{\alpha}$, implying either $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{R}i$. Moreover, $\alpha \neq 0$ since the contrary would imply $|\beta|^2 = -1$, which is an obvious contradiction. If we assume that $\alpha \in \mathbb{R}i$, we see that the term $\alpha^2 - |\beta|^2$ is negative, hence deriving a contradiction. Now if $\alpha \in \mathbb{R}$, we see that $\beta = 0$, since $a\beta + \overline{\alpha}\beta = 2\alpha\beta \overset{!}{=} 0$. Moreover, we see that $\alpha = \pm 1$ since $\alpha^2 - |\beta|^2 = \alpha^2 \overset{!}{=} 1$.

□

Exercise 2. Let X be the nodal curve given by the equation $y^2 = x^3 + x^2$.

1. Find a surjective (polynomial) map $\mathbb{A}^1 \to X$.
2. Let $P \in X$ be the singular point. Show that the completed local ring $\widehat{\mathcal{O}}_{X,P}$ of X at P is isomorphic (as a $\mathbb{C}$-algebra) to $\mathbb{C}[\![x, y]\!]/(xy)$.

Proof. 1. We take the polynomial map

$$\begin{aligned}\mathbb{A}^1 &\to X \\ (t) &\mapsto (t^2 - 1, t(t^2 - 1))\end{aligned}$$

2. We have an isomorphism

$$\widehat{\mathcal{O}}_{X,P} \cong \mathbb{C}[\![x, y]\!]/(y^2 - x^3 - x^2).$$

Now we note that we have the element $\sqrt{1+x} := 1 + \frac{1}{2}x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2k-3)!}{2^{2k-2}k!(k-2)!}x^k \in \mathbb{C}[\![x, y]\!]$ and hence there is a factorization $y^2 - x^3 - x^2 = (y - x\sqrt{1+x})(y + x\sqrt{1+x})$. Now we take the map

$$\begin{aligned}\mathbb{C}[\![x, y]\!] &\to \mathbb{C}[\![x, y]\!] \\ x &\mapsto x\sqrt{1+x} \\ y &\mapsto y\end{aligned}$$

which induces an isomorphism

$$\mathbb{C}[\![x, y]\!]/(y^2 - x^2\sqrt{1+x}^2) \cong \mathbb{C}[\![x, y]\!]/(y^2 - x^2).$$

Finally, the map

$$
\begin{aligned}
\mathbb{C}[\![u,v]\!] &\to \mathbb{C}[\![x,y]\!] \\
u &\mapsto x-y \\
v &\mapsto x+y
\end{aligned}
$$

induces an isomorphism

$$\mathbb{C}[\![x,y]\!]/(y^2-x^2) \cong \mathbb{C}[\![u,v]\!]/(uv).$$

□

Exercise 3. Let $n \geq 2$ and let ζ be a primitive complex root of unity of order n. Let $X = \mathbb{A}^2/G$, where $G = \langle \gamma \rangle \cong \mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{A}^2$ via $\gamma.x = \zeta x, \gamma.y = \zeta^{-1}y$. Let $P = (0,0) \in X$ be the singular point. Show that

$$\widehat{\mathcal{O}}_{X,P} = \mathbb{C}[\![x,y]\!]^G \cong \mathbb{C}[\![u,v,w]\!]/(u^2+v^2+w^n).$$

Proof. A simple comparison of coordinates shows that

$$\mathbb{C}[\![x,y]\!]^G = \mathbb{C}[\![x^n, y^n, xy]\!].$$

We now claim that there is an isomorphism

$$\mathbb{C}[\![x^n, y^n, xy]\!] \cong [\![x,y,z]\!]/(xy-z^n).$$

We take the surjection

$$\varphi\colon \mathbb{C}[\![u,v,w]\!] \twoheadrightarrow \mathbb{C}[\![x^n,y^n,xy]\!].$$

It is obvious, that $(uv - w^n) \subset \ker(\varphi)$. We now want to show the other inclusion. Let $f = \sum_{i,j,k\geq 0} a_{ijk}u^iv^jw^k \in \ker(\varphi)$. Subtracting elements of $(uv-w^n)$ from f, we may assume that $a_{ijk} = 0$ unless $0 \leq k < n$ (here we use implicitly that the ideal is closed). Then

$$0 = \varphi(f) = \sum_{i,j,k\geq 0} a_{ijk}x^{ni+k}y^{nj+k} = \sum_{i',j'\geq 0}\left(\sum_{I(i',j')} a_{ijk}\right)x^{i'}y^{j'}$$

where

$$I(i',j') \coloneqq \{(i,j,k) \mid ni+k = i'nj+k = j', a_{ijk} \neq 0\}.$$

Now we see that for all i', j', $\sum_{I(i',j')} a_{ijk} = 0$. We claim that $f = 0$. Indeed, if $I(i',j') \neq \emptyset$, then $i' - j' = n(i-j)$, so $i' \equiv j'$ modulo n. Moreover, as $0 \leq k \leq n-1$, $k = i' - n \cdot \left\lfloor \frac{i'}{n} \right\rfloor = j' - n \cdot \left\lfloor \frac{j'}{n} \right\rfloor$ is uniquely dtermined by i', j'. Also, $i = \frac{i'-k}{n}, j = \frac{j'-k}{n}$. This shows that $I(i',j')$ either contains one element or is empty. But $I(i',j')$ containing one element is a contradiction to the assumption that $\sum_{I(i',j')} a_{ijk} = 0$. This shows that $I(i',j') = \emptyset$ for all i', j' and hence $f = 0$. Finally, the isomorphism

$$
\begin{aligned}
\mathbb{C}[\![x,y,z]\!] &\to \mathbb{C}[\![u,v,w]\!] \\
x &\mapsto -u+iv \\
y &\mapsto -u-iv \\
z &\mapsto w
\end{aligned}
$$

induces an isomorphism

$$\mathbb{C}[\![x,y,z]\!]/(xy-z^n) \cong \mathbb{C}[\![u,v,w]\!]/(u^2+v^2+w^n).$$

□

Exercise 4. Let $n \geq 2$. Let $G = \mathbb{D}_n$ be the binary dihedral group acting on $\mathbb{C}^2$ (as in the lecture). Let $X = \mathbb{C}^2/G$ and let $P = (0,0) \in X$ be the singular point. Show that

$$\widehat{\mathcal{O}}_{X,P} = \mathbb{C}[\![x,y]\!]^G \cong \mathbb{C}[\![u,v,w]\!]/(u^{n+1}+uv^2+w^2).$$

Proof. The binary dihedral group acts as

$$\sigma\colon \begin{cases} x & \mapsto \zeta x \\ y & \mapsto \zeta^{-1} y \end{cases} \qquad \tau\colon \begin{cases} x & \mapsto -y \\ y & \mapsto x \end{cases}$$

where ζ is a primitive $2n$-th root of unity. Now, one can verify that the set of invariant monomials is

$$F = x^{2n} + y^{2n}, H = xy(x^{2n} - y^{2n}), I = x^2y^2.$$

They satisfy the relation

$$H^2 = x^2y^2(x^{4n} + y^{4n} - 2x^{2n}y^{2n}) = IF^2 - 4I^{n+1}.$$

We now get isomorphisms

$$\begin{aligned} \widehat{\mathcal{O}}_{X,P} = \mathbb{C}[\![x,y]\!]^G &= \mathbb{C}[\![x^{2n} + y^{2n}, xy(x^{2n} - y^{2n}), x^2y^2]\!] \\ &\overset{(*)}{\cong} \mathbb{C}[\![H, I, F]\!]/(H^2 - IF^2 + 4I^{n+1}) \\ &\cong \mathbb{C}[\![u, v, w]\!]/(u^{n+1} + uv^2 + w^2). \end{aligned}$$

The isomorphism (*) can either be seen by an explicit calculation as in exercise 3 or using the fact that a surjective ringhomomorphism $A \to B$ is an isomorphism, if $\dim A = \dim B < \infty$ and A is an integral domain. $\square$

Exercise 5. Let $X = \left\{y^2 = x^3 + x^2\right\} \subseteq \mathbb{A}^2$ be the nodal curve. Let $P \in X$ be the origin. Let $\mathrm{BL}_P(\mathbb{A}^2) = \left\{(x,y), [u:v] \in \mathbb{A}^2 \times \mathbb{P}^1 \,\middle|\, xv = yu\right\}$ be the blowup of $\mathbb{A}^2$ at P. Compute the strict transform $\widetilde{X}$ of X inside $\mathrm{BL}_P(\mathbb{A}^2)$.

Proof. We take the map $\pi\colon \mathrm{BL}_P(\mathbb{A}^2) \to \mathbb{A}^2$ and take the restriction π_v to the locus where $v \neq 0$. Then:

$$\pi_v\colon \{(x,y), [u:1] \,|\, x = yu\} \to \mathbb{A}^2$$

and

$$\begin{aligned} \pi_v^{-1}(X) &= \left\{(x,y), [u:1] \,\middle|\, x = yu, y^2 = x^3 + x^2\right\} \\ &= \left\{(yu,y), [u:1] \,\middle|\, y^2 = y^3u^3 + y^2u^2\right\} \\ &= \left\{y^2 = 0\right\} \cup \underbrace{\left\{1 = yu^3 + u^2\right\}}_{\text{part of } \widetilde{X} \text{ on } \{v \neq 0\}}. \end{aligned}$$

We then get

$$\widetilde{X} \cap \{v \neq 0\} = \left\{1 = yu^3 + u^2\right\}$$

and

$$\widetilde{X} \cap \{v \neq 0\} \cap E = \left\{1 = u^2\right\} = \{(0,0), [\pm 1 : 1]\}.$$

$\square$

Exercise 6. Compute the blow-ups and the dual graphs of the Du Val singularities of type A_2, A_3 and D_4.

Proof. We take the blow up

$$\mathrm{BL}_P(\mathbb{A}^3) = \{(x,y,z), [u:v:w] \,|\, xv = yu, xw = zu, yw = zv\}.$$

We recall the the charts:

$$\begin{aligned} \pi_u\colon \{(x,y,z), [1:v:w] \,|\, xv = y, xw = z, yw = zv\} &\to \mathbb{A}^3 \\ \pi_v\colon \{(x,y,z), [u:1:w] \,|\, x = yu, xw = zu, yw = z\} &\to \mathbb{A}^3 \\ \pi_w\colon \{(x,y,z), [u:v:1] \,|\, xv = yu, x = zu, y = zv\} &\to \mathbb{A}^3 \end{aligned}$$

We now take $X_1 = V(x^2 + y^2 + z^3) \subset \mathbb{A}^3$ and get

$$\begin{aligned}
\pi_u^{-1}(X_1) &= \big\{(x, y, z), [1 : v : w] \,\big|\, x^2 + y^2 + z^3 = 0, xv = y, xw = z, yw = zv\big\} \\
&= \big\{(x, xv, xw), [1 : v : w] \,\big|\, x^2 + x^2v^2 + x^3w^3 = 0\big\} \\
&= \big\{x^2 = 0\big\} \cup \big\{1 + v^2 + xw^3 = 0\big\} \\
\pi_v^{-1}(X_1) &= \big\{(x, y, z), [u : 1 : w] \,\big|\, x^2 + y^2 + z^3 = 0, x = yu, xw = zu, yw = z\big\} \\
&= \big\{(yu, y, yw), [u : 1 : w] \,\big|\, y^2u^2 + y^2 + y^3w^3 = 0\big\} \\
&= \big\{y^2 = 0\big\} \cup \big\{u^2 + 1 + yw^3\big\} \\
\pi_w^{-1}(X_1) &= \big\{(x, y, z), [u : v : 1] \,\big|\, x^2 + y^2 + z^3 = 0, xv = yu, x = zu, y = zv\big\} \\
&= \big\{(zu, zv, z), [u : v : 1] \,\big|\, z^2u^2 + z^2v^2 + z^3 = 0\big\} \\
&= \big\{z^2 = 0\big\} \cup \big\{u^2 + v^2 + z = 0\big\}
\end{aligned}$$

We use the Jacobian criterion for the first chart to get the system

$$\begin{cases} w^3 & = 0 \\ 2v & = 0 \\ 3xw^2 & = 0 \end{cases}$$

which we see has no solutions, so this chart is smooth. Using the Jacobian criterion for the second chart gives the same result. Using the Jacobian criterion for the third chart gives us

$$\begin{cases} 2u & = 0 \\ 2v & = 0 \\ 1 & = 0 \end{cases}$$

which also has no solution, so this is also smooth. We now take the exceptional divisor

$$E_0 = \widetilde{X}_1 \cap E.$$

Its intersections with the first two charts are of the form $\{[1 : \pm i : w]\}$, while its intersection with the third chart is of the form $\{[u : \pm iu : 1]\}$. This shows there are two copies of $\mathbb{P}^1$, intersecting in the point $[0 : 0 : 1]$ and hence the Dynkin diagram is of the form

A_2 $\bullet\!\!-\!\!\bullet$

We now take $X_2 = V(x^2 + y^2 + z^4) \subset \mathbb{A}^3$ and get

$$\begin{aligned}
\pi_u^{-1}(X_2) &= \big\{(x, y, z), [1 : v : w] \,\big|\, x^2 + y^2 + z^4 = 0, xv = y, xw = z, yw = zv\big\} \\
&= \big\{(x, xv, xw), [1 : v : w] \,\big|\, x^2 + x^2v^2 + x^4w^4 = 0\big\} \\
&= \big\{x^2 = 0\big\} \cup \big\{1 + v^2 + x^2w^4 = 0\big\} \\
\pi_v^{-1}(X_2) &= \big\{(x, y, z), [u : 1 : w] \,\big|\, x^2 + y^2 + z^4 = 0, x = yu, xw = zu, yw = z\big\} \\
&= \big\{(yu, y, yw), [u : 1 : w] \,\big|\, y^2u^2 + y^2 + y^4w^4 = 0\big\} \\
&= \big\{y^2 = 0\big\} \cup \big\{u^2 + 1 + y^2w^4\big\} \\
\pi_w^{-1}(X_2) &= \big\{(x, y, z), [u : v : 1] \,\big|\, x^2 + y^2 + z^4 = 0, xv = yu, x = zu, y = zv\big\} \\
&= \big\{(zu, zv, z), [u : v : 1] \,\big|\, z^2u^2 + z^2v^2 + z^4 = 0\big\} \\
&= \big\{z^2 = 0\big\} \cup \big\{u^2 + v^2 + z^2 = 0\big\}
\end{aligned}$$

Using the Jacobian criterion on the first chart gives us

$$\begin{cases} 2xw^4 & = 0 \\ 2v & = 0 \\ 3x^2w^3 & = 0 \end{cases}$$

which has no solutions. The same applies to the second chart. For the third chart, we get

$$\begin{cases} 2u & = 0 \\ 2v & = 0 \\ 2z & = 0 \end{cases}$$

which shows that there is a unique singular point $u = v = z = 0$, or in global coordinates $(0, 0, 0), [0 : 0 : 1]$. By calculating the exceptional fibre, we get the intersection with first and second chart $\{[1 : \pm i : w]\}$ and the intersection with the third chart $\{[u : \pm iu : 1]\}$, giving us two copies of $\mathbb{P}^1$ as above.

We now blow this up again (for brevity, we again use the coordinates x, y, z): We blow up the surface $X_3 = V(x^2 + y^2 + z^2) \subset \mathbb{A}^3$ with exceptional divisor E_0 having the coordinates $z = 0$ and $x = \pm iy$.

$$\begin{aligned}
\pi_u^{-1}(X_3) &= \big\{(x, y, z), [1 : v : w] \,\big|\, x^2 + y^2 + z^2 = 0, xv = y, xw = z, yw = zv\big\} \\
&= \big\{(x, xv, xw), [1 : v : w] \,\big|\, x^2 + x^2v^2 + x^2w^2 = 0\big\} \\
&= \big\{x^2 = 0\big\} \cup \big\{1 + v^2 + w^2 = 0\big\} \\
\pi_v^{-1}(X_3) &= \big\{(x, y, z), [u : 1 : w] \,\big|\, x^2 + y^2 + z^2 = 0, x = yu, xw = zu, yw = z\big\} \\
&= \big\{(yu, y, yw), [u : 1 : w] \,\big|\, y^2u^2 + y^2 + y^2w^2 = 0\big\} \\
&= \big\{y^2 = 0\big\} \cup \big\{u^2 + 1 + w^2\big\} \\
\pi_w^{-1}(X_3) &= \big\{(x, y, z), [u : v : 1] \,\big|\, x^2 + y^2 + z^2 = 0, xv = yu, x = zu, y = zv\big\} \\
&= \big\{(zu, zv, z), [u : v : 1] \,\big|\, z^2u^2 + z^2v^2 + z^2 = 0\big\} \\
&= \big\{z^2 = 0\big\} \cup \big\{u^2 + v^2 + 1 = 0\big\}
\end{aligned}$$

The Jacobian criterion gives for the first chart

$$\begin{cases} 2u &= 0 \\ 2v &= 0 \end{cases}$$

which has no solution and the same applies to the other charts, showing that this are no singular points. Since all three charts of $\widetilde{X}_3$ are symmetric, the exceptional divisor E is smooth and hence $E \cong \mathbb{P}^1$. This gives us the dual graph

A_3

Now we take $X_4 = V(x^2 + y^2z + z^3) \subseteq \mathbb{A}^3$ and get

$$\begin{aligned}
\pi_u^{-1}(X_4) &= \big\{(x, y, z), [1 : v : w] \,\big|\, x^2 + y^2z + z^3 = 0, xv = y, xw = z, yw = zv\big\} \\
&= \big\{(x, xv, xw), [1 : v : w] \,\big|\, x^2 + x^2v^2xw + x^3w^3\big\} \\
&= \big\{x^2 = 0\big\} \cup \big\{1 + xv^2w + xw^3 = 0\big\} \\
\pi_v^{-1}(X_4) &= \big\{(x, y, z), [u : 1 : w] \,\big|\, x^2 + y^2z + z^3 = 0, x = yu, xw = zu, yw = z\big\} \\
&= \big\{(yu, y, yw), [u : 1 : w] \,\big|\, y^2u^2 + y^2yw + y^3w^3 = 0\big\} \\
&= \big\{y^2 = 0\big\} \cup \big\{u^2 + yw + yw^3\big\} \\
\pi_w^{-1}(X_4) &= \big\{(x, y, z), [u : v : 1] \,\big|\, x^2 + y^2z + z^3 = 0, xv = yu, x = zu, y = zv\big\} \\
&= \big\{(zu, zv, z), [u : v : 1] \,\big|\, z^2u^2 + z^2v^2z + z^3 = 0\big\} \\
&= \big\{z^2 = 0\big\} \cup \big\{u^2 + v^2z + z = 0\big\}
\end{aligned}$$

The Jacobian criterion gives for the first chart

$$\begin{cases} v^2w + w^3 &= 0 \\ 2xvw &= 0 \\ xv^2 + 3xw^2 &= 0 \end{cases}$$

A quick investigation shows that this has no solution. For the second chart we get

$$\begin{cases} 2u = 0 \\ w + w^3 = 0 \\ y + 3w^2y = 0 \end{cases}$$

This has three solutions $\{(0, 0, 0), (0, 0, i), (0, 0, -i)\}$. For the third chart, we get

$$\begin{cases} 2u = 0 \\ 2vz = 0 \\ v^2 + 1 = 0 \end{cases}$$

which has the solutions $(0,i,0),(0,-i,0)$ (in global coordinates, those are already included in the above!). Calculating the exceptional fibre, we get an empty intersection with the first chart, the intersection with the second chart $\{[0:1:w]\}$ and the intersection with the third chart $\{[0:v:1]\}$, giving us the exceptional divisor $E \cong \mathbb{P}^1$. Now, we blow up at the first singular point $(0,0,0)[0:1:0]$. We have $X_5 = V(x^2+yz+yz^3)$ and get

$$\begin{aligned}
\pi_u^{-1}(X_5) &= \big\{(x,y,z),[1:v:w] \,\big|\, x^2+yz+yz^3=0, xv=y, xw=z, yw=zv\big\} \\
&= \big\{(x,xv,xw),[1:v:w] \,\big|\, x^2+xvxw+xvx^3w^3=0\big\} \\
&= \big\{x^2=0\big\} \cup \big\{1+vw+x^2vw^3=0\big\} \\
\pi_v^{-1}(X_5) &= \big\{(x,y,z),[u:1:w] \,\big|\, x^2+yz+yz^3=0, x=yu, xw=zu, yw=z\big\} \\
&= \big\{(yu,y,yw),[u:1:w] \,\big|\, y^2u^2+yyw+yy^3w^3=0\big\} \\
&= \big\{y^2=0\big\} \cup \big\{u^2+w+y^2w^3\big\} \\
\pi_w^{-1}(X_5) &= \big\{(x,y,z),[u:v:1] \,\big|\, x^2+yz+yz^3=0, xv=yu, x=zu, y=zv\big\} \\
&= \big\{(zu,zv,z),[u:v:1] \,\big|\, z^2u^2+zvz+zvz^3=0\big\} \\
&= \big\{z^2=0\big\} \cup \big\{u^2+v+z^2v\big\}
\end{aligned}$$

The Jacobian criterion gives for the first chart

$$\begin{cases} 2xvw^3 &= 0 \\ w+x^2w^3 &= 0 \\ v+3w^2x^2v &= 0 \end{cases}$$

which has no solution. For the second chart, we get

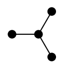
$$\begin{cases} 2u &= 0 \\ 1+3w^2y^2 &= 0 \\ 2yw^3 &= 0 \end{cases}$$

which also has no solutions. For the third chart, we get

$$\begin{cases} 2u &= 0 \\ 1+z^2 &= 0 \\ 2zv &= 0 \end{cases}$$

which has two singularities $(0,0,i),(0,0,-i)$ (which do not lie in the exceptional divisor and come from the above singularities!). We calculate the exceptional divisor: The intersection with the first chart is $\big\{[1:v^2:v] \,\big|\, v \neq 0\big\}$, the intersection with the second chart is $\big\{[u:1:-u^2]\big\}$ and the intersection with the third chart is $\big\{[[u:-u^2:1]]\big\}$, giving us $\mathbb{P}^1$.

Changing coordinates, we see that the other two singularities are of this form as well, giving us the dual graph

D_4 

. $\square$

Exercise 7. Prove that the singularities of type A_3 (given by $\big\{x^2+y^2+z^4=0\big\}$) and D_3 (given by $\big\{x^2+y^2z+z^2=0\big\}$) are isomorphic.

Proof. There is an isomorphism

$$\begin{aligned}
\mathbb{C}[\![x,y,z]\!] &\to \mathbb{C}[\![x,y,z]\!] \\
x &\mapsto x \\
y &\mapsto z+\frac{1}{2}y^2 \\
z &\mapsto \frac{i+1}{2}y
\end{aligned}$$

which induces an isomorphism

$$\mathbb{C}[\![x,y,z]\!]/(x^2+y^2z+z^2) \cong \mathbb{C}[\![x,y,z]\!]/(x^2+y^2+z^4).$$

Alternatively, we can use that $\mathbb{D}_1$ (which is of type D_3) and the group of type A_3 are conjugated. $\square$